The Order Parameter of the Chiral Potts Model

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An outstanding problem in statistical mechanics is the order parameter of the chiral Potts model. An elegant conjecture for this was made in 1983. It has since been successfully tested against series expansions, but there is as yet no proof of the conjecture. Here we show that if one makes a certain analyticity assumption similar to that used to derive the free energy, then one can indeed verify the conjecture. The method is based on the "broken rapidity line" approach pioneered by Jimbo *et al.* (*J. Phys. A* **26**:2199–2210 (1993).).

1. INTRODUCTION

The chiral Potts model was originally formulated as an N-state one-dimensional quantum hamiltonian,[1,2] and then as a two-dimensional classical lattice model in statistical mechanics.[3–5] It satisfies the star-triangle relations. The free energy was first obtained for the infinite lattice using the invariance properties of the free energy and its derivatives.[6] Then in 1990 the functional transfer matrix relations of Bazhanov and Stroganov[7] were used to calculate the free energy more explicitly as a double integral.[8–10]

The chiral Potts model is a system of spins living on the sites of a planar lattice, usually (but not necessarily[11]) taken to be the square lattice $\mathcal{L}$. Each spin takes one of N possible states, labelled $0, \dots, N-1$ and interacts with its neighbours. If the spin at site i is σ_i, then the interaction between spins on adjacent sites i and j depends on σ_i, σ_j only via their difference $\sigma_i - \sigma_j \pmod{N}$.

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We shall define the interactions in Section 2. Here we merely note that they depend on a temperature-like parameter k', which is small at low temperatures and large at high temperatures. For $k' < 1$, a related parameter is

$$k = \sqrt{(1 - k'^2)}. \tag{1}$$

Let $\omega = e^{2\pi i/N}$ and let a be a spin deep inside the lattice. Define

$$\mathcal{M}_r = \langle \omega^{ra} \rangle, \tag{2}$$

as the average value of ω^{ra}, for $r = 0, \dots, N$. Suppose one fixes the boundary spins to be zero and allows the lattice to become infinitely large, a remaining near the centre. Then for sufficiently high temperatures (weak enough interactions) the boundary conditions will become irrelevant, all values of a will be equally likely, and $\mathcal{M}_r$ will be exactly zero for $r = 1, \dots, N-1$.

However, the system displays ferromagnetic order. There is a critical temperature T_c below which the boundary conditions remain relevant even for an infinitely large lattice. Then $\mathcal{M}_1, \dots, \mathcal{M}_{N-1}$ are non-zero. They can be thought of as spontaneous magnetizations or order parameters. We expect $\mathcal{M}_r$ to vanish as $T \to T_c$, being then proportional to

$$(1 - T/T_c)^\beta$$

the index β being known as a critical exponent (dependent on r).

In fact the critical point is when $k' = 1$ and $k = 0$, so the ferromagnetic region is when $0 < k, k' < 1$. For $N = 2$ the chiral Potts model reduces to the Ising model. From the exact results of Yang[12] and Onsager[13] we know that then $\mathcal{M}_1 = k^{1/4}$. Hence $\mathcal{M}_1 = (1 - k'^2)^{1/8}$ and the critical exponent β is $1/8$.

In 1983 Howes et al.[1] considered the case $N = 3$ and evaluated $\mathcal{M}_1, \mathcal{M}_2$ to order k'^{13} in a series expansion in powers of k'. They found their series fitted the formula $\mathcal{M}_1 = \mathcal{M}_2 = k^{2/9}$, giving $\beta = 1/9$. Later, Henkel and Lacki[14] expanded $\sum \mathcal{M}_r$ for general N to order k'^6.

In 1989 Albertini et al.[15] expanded the general N case to order k'^5, and found that all the results were consistent with the remarkably simple and elegant conjecture

$$\mathcal{M}_r = k^{r(N-r)/N^2}, \quad 0 \leqslant r \leqslant N, \tag{3}$$

(see also ref. 16), giving $\beta = r(N - r)/2N^2$. Baxter[11] used finite corner transfer matrices to expand $\mathcal{M}_r$ for $N = 3$ to order k'^{14}, and again found the results were consistent with the conjecture (3).

So for the last 15 years or more there have been conjectures for the order parameters of the chiral Potts model. However, the author knows of no derivation of any of these for $N > 2$. One can contrast this with the situation for the Ising model, where it was five years from the time Onsager calculated the free energy to when at a conference in Florence in 1949 he announced the formula for $\mathcal{M}_1$, and three years from then until Yang published a proof. (The actual calculation took Yang six months of work off and on (ref. 17, p. 12).)

The difficulty has been that unlike most other solvable models, the chiral Potts model does *not* have the "rapidity difference property". For this reason the infinite-size "corner transfer method"[18] for exactly calculating the order parameters fails. An alternative method was developed in 1993 by Jimbo *et al.*[19] It relies on breaking one of the rapidity lines of the lattice, giving one half-line a different rapidity value from the other. The author applied this method to the chiral Potts in 1998[20] and wrote down functional relations for the resulting generalized order parameters $G_{pq}(r)$. They are functions of the rapidities p, q of the two half-lines.

Again there was a difficulty. The relations by themselves do not completely define the $G_{pq}(r)$. One also needs information as to the analyticity properties of the functions.[2] In this respect the functional relations are similar to the "inversion relations" for the infinite-lattice free energy.[21,22]

The calculation of the free energy of the chiral Potts model[9,10,22] proceeds in two stages. First one considers a related "$\tau_2(t_p)$" model.[23] This is intimately connected with the superintegrable case of the chiral Potts model.[24] It is much simpler than the chiral Potts model in that its Boltzmann weights depend on the horizontal rapidity p only via a single parameter t_p, and are linear in t_p. Its row-to-row transfer matrix is the product of two chiral Potts transfer matrices, one with horizontal rapidity p, the other with a related rapidity $\bar{p}(0, 2)$ defined in Eq. (12) of Section 3.

For a finite lattice, the partition function Z of the $\tau_2(t_p)$ model is therefore a polynomial in t_p. The free energy is the logarithm of $Z^{1/M}$, where M is the number of sites of the lattice, evaluated in the thermodynamic limit when the lattice becomes infinitely big. This limiting function

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[2]The example I like to quote is the relation $f(z + 1) = f(z)$. By itself this merely says that $f(z)$ is periodic. However, if one can also show that $f(z)$ is analytic and bounded in the strip $0 \leqslant \Re(z) \leqslant 1$, then it follows that $f(z)$ is analytic everywhere and bounded, so by Liouville's theorem it is a constant.

of course may have singularities in the complex t_p plane. *A priori*, one might expect it to have N branch cuts, each running though one of the N roots of unity. However, one can argue that in fact it only has one such cut. As a result the free energy (i.e., the maximum eigenvalue of the transfer matrix) can be calculated by a Wiener–Hopf factorization.

The second stage is to factor this free energy to obtain that of the chiral Potts model.

It turns out that the first stage of this process can be applied to the generalized order parameter function $G_{pq}(r)$, provided we take q, p to be related by Eq. (62) of Section 6.

We present the working in the following sections. We do not need to calculate $G_{pq}(r)$ for general p, q and we do not do so.

We define the model in Section 2, and the function $G_{pq}(r)$ in Section 4. We also present the functional relations satisfied by $G_{pq}(r)$, but in fact we hardly use them. It is the analyticity properties that hold the key to calculating the $G_{pq}(r)$, and most of the discussion in Sections 3–6 is concerned with presenting evidence for our assumptions regarding these properties.

In particular, we show that when p, q satisfy (62), the $G_{pq}(r)$ can be expressed in terms of $\tau_2(t_p)$ Boltzmann weights $U(a, b, c, d)$ that are linear in t_p. We argue that $G_{pq}(r)$ is therefore like the free energy of the $\tau_2(t_p)$ model, in that it has at most one branch cut in the t_p-plane, rather than the N cuts that one might expect.

We give our precise assumptions in Section 7. Then in Section 8 we use them to obtain $G_{pq}(r)$ by a Wiener–Hopf factorization in very much the same way as one calculates the $\tau_2(t_p)$ free energy. The desired formula (3) follows immediately. We also present an alternative method, looking at the product of N such functions $G_{pq}(r)$, that avoids the need for a Wiener–Hopf factorization.

In Section 9, we briefly discuss some other special cases (analogues of the $\tau_j(t_p)$ models for $j = 3, \ldots, N$) that may be tractable, and make a conjecture for the form of $G_{pq}(r)$ for some of these cases.

2. CHIRAL POTTS MODEL

We use the notation of refs. 5, 8, 20. Let k, k' be two real variables in the range $(0, 1)$, satisfying (1). Consider four parameters a_p, b_p, c_p, d_p satisfying the homogeneous relations

$$
\begin{aligned}
a_p^N + k' b_p^N &= k d_p^N, \\
k' a_p^N + b_p^N &= k c_p^N.
\end{aligned}
\tag{4}
$$

Let p denote the set $\{a_p, b_p, c_p, d_p\}$, or rather their ratios, since all the equations we shall write involve a_p, b_p, c_p, d_p only via their ratios. Similarly q denotes the set $\{a_q, b_q, c_q, d_q\}$ satisfying the relations (4) (with p replaced by q). We call p and q "rapidity" variables.

Define Boltzmann weight functions $W_{pq}(n)$, $\bar{W}_{pq}(n)$ by

$$
\begin{aligned}
W_{pq}(n) &= \prod_{j=1}^{n} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j}, \\
\bar{W}_{pq}(n) &= \prod_{j=1}^{n} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j},
\end{aligned}
\tag{5}
$$

where

$$
\omega = e^{2\pi i/N}.
$$

Then the conditions (4) ensure that

$$
W_{pq}(n+N) = W_{pq}(n), \qquad \bar{W}_{pq}(n+N) = \bar{W}_{pq}(n),
$$

so the functions are periodic of period N. Note that

$$
W_{pq}(0) = \bar{W}_{pq}(0) = 1.
$$

Define related parameters

$$
x_p = a_p/d_p, \qquad y_p = b_p/c_p, \qquad t_p = x_p y_p, \qquad \mu_p = d_p/c_p.
\tag{6}
$$

They satisfy

$$
x_p^N + y_p^N = k(1 + x_p^N y_p^N), \qquad kx_p^N = 1 - k'/\mu_p^N, \qquad ky_p^N = 1 - k'\mu_p^N,
\tag{7}
$$

$$
k^2 t_p^N = (1 - k'\mu_p^N)(1 - k'/\mu_p^N).
$$

We can also write (5) as

$$
W_{pq}(n) = (\mu_p/\mu_q)^n \prod_{j=1}^{n} \frac{y_q - \omega^j x_p}{y_p - \omega^j x_q},
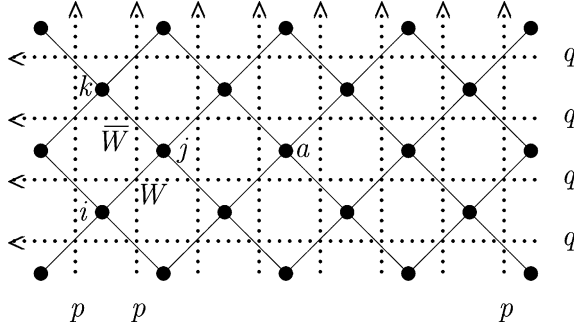\tag{8a}
$$

Fig. 1.   The square lattice $\mathcal{L}$ of solid lines and circles, showing the central spin a and the dotted directed rapidity lines.

$$\bar{W}_{pq}(n) = (\mu_p \mu_q)^n \prod_{j=1}^{n} \frac{\omega x_p - \omega^j x_q}{y_q - \omega^j y_p}. \tag{8b}$$

Now consider the square lattice $\mathcal{L}$, drawn diagonally as in Fig. 1, with a total of M sites. On each site i place a spin σ_i, which can take any one of the N values $0, 1, \ldots, N-1$. With each SW–NE edge (i, j) (with i below j) associate an edge weight $W_{pq}(\sigma_i - \sigma_j)$. Similarly, with each SW–NE edge (j, k) (j below k), associate an edge weight $\bar{W}_{pq}(\sigma_j - \sigma_k)$. Then the partition function is

$$Z = \sum_{\sigma} \prod W_{pq}(\sigma_i - \sigma_j) \prod \bar{W}_{pq}(\sigma_j - \sigma_k) \tag{9}$$

the products being over all edges of each type, and the sum over all N^M values of the M spins. We expect the partition function per site

$$\kappa = Z^{1/M},$$

to tend to a unique limit as the lattice becomes large in both directions.

Let a be a spin on a site near the centre of the lattice, as in Fig. 1, and let $f(a)$ be any function thereof. Then the thermodynamic average of $f(a)$ is

$$\langle f(a) \rangle = Z^{-1} \sum_{\sigma} f(a) \prod W_{pq}(\sigma_i - \sigma_j) \prod \bar{W}_{pq}(\sigma_j - \sigma_k). \tag{10}$$

we expect this to also tend to a limit as the lattice becomes large.

We have also shown in Fig. 1 the vertical and horizontal "rapidity lines". Each edge of $\mathcal{L}$ passes through the intersection of two rapidity lines.

We could allow each vertical (horizontal) rapidity line α to have a different rapidity p_α (q_α). If an edge of $\mathcal{L}$ lies on lines with rapidities p_α, q_β, then the Boltzmann weight function of that edge is to be taken as $W_{pq}(n)$ or $\bar{W}_{pq}(n)$, with $p = p_\alpha$ and $q = q_\beta$.

The weight functions $W_{pq}(n)$, $\bar{W}_{pq}(n)$ satisfy the star–triangle relation.[5] For this reason we are free to move the rapidity lines around in the plane, in particular to interchange two vertical or two horizontal rapidity lines.[25] So long as no rapidity line crosses the site with spin a while making such rearrangements, the average $\langle f(a) \rangle$ is *unchanged* by the rearrangement.[3]

All of the rapidity lines shown in Fig. 1 are "full", in the sense that they extend without break from one boundary to another. We can move any such line away from the central site to infinity, where we do not expect it to contribute to $\langle f(a) \rangle$. Hence $\langle f(a) \rangle$ *must be independent of all the full-line p and q rapidities*.

It can still depend on k or k', which play the role of universal constants, the same for all sites of $\mathcal{L}$. As we mentioned in the introduction, it has been conjectured that

$$\mathcal{M}_r = \langle \omega^{ra} \rangle = k^{r(N-r)/N^2}, \quad 0 \leqslant r \leqslant N. \tag{11}$$

The aim of this paper is to verify this conjecture, subject to a plausible analyticity assumption.

3. $\tau_2\ (t_p)$ MODEL

One of the difficulties of the chiral Potts model is the multi-valued nature of the relations between x_p, y_p, μ_p. Every such relation involves taking an Nth root. For $N = 2$ the model reduces to the Ising model. In this case there is a simple uniformizing substitution whereby all variables can be written as single-valued Jacobi elliptic functions of another variable u_p,[22] Appendix B. For higher values of N no such substitution is known. It is therefore significant that the chiral Potts model can be related to "superintegrable" or "$\tau_j(t_p)$" models, where the dependence of Boltzmann weights on the horizontal rapidity p is simple: they are explicit polynomials in the single variable t_p. (They still involve all the vertical rapidity variables x_v, y_v, μ_v.)

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[3]Subject to boundary conditions: here we are primarily interested in the infinite lattice, where we expect the boundary conditions to have no effect on the rearrangements we consider.

The key equations are given in ref. 8. Let us change notation and use the symbols v or v' for the vertical rapidities, p, q for the rapidities of the lower and upper rows. Thus we replace the p, p', q, r of ref. 8 by v, v', p, q.

For generality, we allow v and v' to be different in this section. In later sections we take $v' = v$ and all vertical rapidities to be the same. p and q will play the role of variables in the functional relations we discuss, while we shall regard v as a constant.

Consider two rows of edges of $\mathcal{L}$. Let the horizontal rapidity of the lower row be p, and of the upper row

$$q = \bar{p}(k, \ell), \tag{12}$$

by which we mean

$$x_q = \omega^k y_p, \quad y_q = \omega^\ell x_p, \quad \mu_q = 1/\mu_p, \tag{13}$$

k, ℓ being integers. We can impose the restriction

$$1 \leqslant k + \ell \leqslant N. \tag{14}$$

If we sum over an intermediate spin between these rows, we construct a combined weight function

$$U(a, b, c, d) = \sum_{g=0}^{N-1} W_{vp}(a-g) \, \bar{W}_{vq}(g-d) \, \bar{W}_{v'p}(b-g) \, W_{v'q}(g-c), \tag{15}$$

depicted in Fig. 2. This can in turn be written as

$$U(a, b, c, d) = N \sum_{n=0}^{N-1} V_{vpq}(a, d; n) \, V_{v'qp}(-c, -b; n), \tag{16}$$

where

$$V_{vpq}(a, d; n) = N^{-1} \sum_{g=0}^{N-1} \omega^{ng} \, W_{vp}(a - g) \, \bar{W}_{vq}(g - d). \tag{17}$$
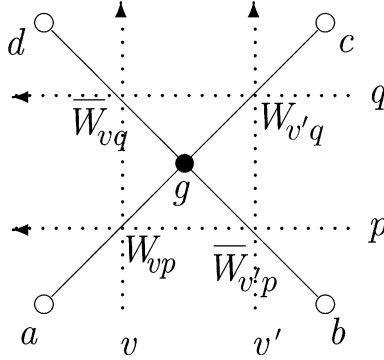
Fig. 2. The combined weight $U(a, b, c, d)$ of four edges, summed over the middle spin g.

Let $\zeta_{k\ell}$ be the set of integers $\{-k, 1-k, \dots, \ell-1\}$, with $k+\ell$ elements. We say that $n \in \zeta_{k\ell}$ if n is equal, modulo N, to one of the elements of $\zeta_{k\ell}$. Then in (3.16) and (3.17) of ref. 8 it is shown that

$$
\begin{aligned}
V_{vpq}(a, d; n) = 0 \quad &\text{if } a-d \in \zeta_{k\ell} \text{ and } n \notin \zeta_{k\ell}, \\
V_{vqp}(-c, -b; n) = 0 \quad &\text{if } n \in \zeta_{k\ell} \text{ and } b-c \notin \zeta_{k\ell}, \\
U(a, b, c, d) = 0 \quad &\text{if } a-d \in \zeta_{k\ell} \text{ and } b-c \notin \zeta_{k\ell}.
\end{aligned}
\tag{18}
$$

Consider a row of adjacent spins $\sigma_1, \dots, \sigma_s$, below a similar row $\sigma_1', \dots, \sigma_s'$, as in Fig. 3, with intervening face weights (15). For the moment ignore the spin g and the dotted lines. The combined weight of the rows is

$$
I(\sigma_1, \dots, \sigma_s') = \prod_{i=1}^{s-1} U(\sigma_i, \sigma_{i+1}, \sigma_{i+1}', \sigma_i')
\tag{19}
$$

and it follows that if $\sigma_1 - \sigma_1' \in \zeta_{k\ell}$, then the weight is zero unless $\sigma_i - \sigma_i' \in \zeta_{k\ell}$ for $i = 1, \dots, s$. In this case we only need $U(a, b, c, d)$ for $a-d$ and $b-c$ in $\zeta_{k\ell}$. From (3.21) to (3.39) of ref. 8, if we define

$$
\begin{aligned}
j = k+\ell, \quad &\alpha = \mathrm{mod}\,(a-d+k, N), \\
m = \mathrm{mod}\,(n+k, N), \quad &\beta = \mathrm{mod}\,(b-c+k, N),
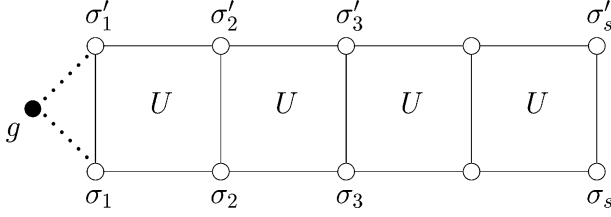\end{aligned}
\tag{20}
$$

Fig. 3. Two rows of spins with face weight functions $U(\sigma_i, \sigma_{i+1}, \sigma'_{i+1}, \sigma'_i)$ in between, and a spin g to the left linked by edge weights to σ_1, σ'_1.

then

$$
\begin{aligned}
V_{vpq}(a, d; n) &= \Omega^{kl}_{vp}\, \omega^{nd-mk}\, (b_p/d_p)^\alpha\, y_p^{-m}\, f_{vp}(j, \alpha, m), \\
V_{v'qp}(-c, -b; n) &= \bar{\Omega}^{kl}_{v'p}\, \bar{h}^{(j)}_{v'p}\, \omega^{kc-mb}(d_p/b_p)^\beta \frac{\eta_{p,j,\beta}}{\eta_{p,j,m}}\, y_p^m\, f_{v'p}(j, \beta, m),
\end{aligned}
\tag{21}
$$

provided $a - d, n, b - c$ are all in $\zeta_{k\ell}$, which means that

$$
0 \leqslant \alpha, m, \beta < j \leqslant N.
\tag{22}
$$

The expressions (21) become less frightening if one groups them into factors of various types:

(i) **Factors independent of** $a, b, c, d, n, \alpha, \beta, m$
These are the factors Ω^{kl}_{vp}, $\bar{\Omega}^{kl}_{v'p}$, $\bar{h}^{(j)}_{v'p}$. They are defined in (3.24) and (3.35) of ref. 8. In this paper, we do not calculate full partition functions, but rather ratios of partition functions such as (34) and (47). These factors cancel out of such ratios, so we can ignore them. Hereinafter we shall simply take $\Omega^{kl}_{vp} = \bar{\Omega}^{kl}_{v'p} = \bar{h}^{(j)}_{v'p} = 1$ in (21).

(ii) **Factors that are powers of** (b_d/d_p)
These are the factors $(b_p/d_p)^\alpha$, $(d_p/b_p)^\beta$. In Fig. 3 each internal vertical edge (σ_j, σ'_j) acquires a weight $(d_p/b_p)^\beta$ from the face on its left, and a factor $(d_p/b_p)^{-\beta}$ from the face on its right, where $\beta = \mathrm{mod}\,(\sigma_j - \sigma'_j + k, N)$. The contributions to the internal edges therefore cancel.

(iii) **Factors** y_p^{-m}, y_p^m
These cancel from the product on the rhs of (15).

(iv) **The** ω, η, f_{vp} **factors**
These remaining factors depend on the rapidity p only via the functions $\eta_{p,j,\alpha}$, $f_{vp}(j, \alpha, m)$. These are defined in (3.26), (3.37) and (3.38) of ref. 8 (where our f_{vp} becomes F_{pq}). They have a vital property: *they are Laurent*

polynomials in t_p, with no other dependence on p. More strongly, $\eta_{p,j,\alpha}$ is proportional to t_p^α, and $f_{vp}(j, \alpha, m)$ is a polynomial in t_p of degree not greater than m and $j - \alpha - 1$, with minimum power not less than zero and $m - \alpha$.

It follows from (16) that $(d_p/b_p)^{\alpha - \beta} U(a, b, c, d)$ is a polynomial in t_p of degree $j - 1$. Hence from (19)

$$I(\sigma_1, \dots, \sigma_s') = (b_p/d_p)^{\lambda - \nu} \mathcal{I}(\sigma_1, \dots, \sigma_s'), \tag{23}$$

where $\mathcal{I}(\sigma_1, \dots, \sigma_s')$ is a polynomial in t_p of degree $(s - 1)(j - 1)$ and $\lambda = \mathrm{mod}\,(\sigma_1 - \sigma_1' + k, N)$, $\nu = \mathrm{mod}\,(\sigma_s - \sigma_s' + k, N)$.

A related quantity that we shall need is

$$J_r(\sigma_1, \dots, \sigma_s') = V_{v'qp}(-\sigma_1', -\sigma_1; -r)\, I(\sigma_1, \dots, \sigma_s'), \tag{24}$$

where r is some integer. This corresponds to including the spin g in Fig. 3, together with weight functions between g and σ_1, σ_1', multiplying by ω^{rg}, and summing over g. The type (ii) factors now cancel from the left-hand edge (σ_1, σ_1'), but a type (iii) factor y_p^m arises from the extra $V_{v'qp}$ term. Using $y_p = t_p/x_p$, it follows that

$$x_p^{m(r)} J_r(\sigma_1, \dots, \sigma_s') = (d_p/b_p)^\nu t_p^{j-1} \mathcal{J}_r(\sigma_1, \dots, \sigma_s'), \tag{25}$$

where $m(r) = \mathrm{mod}\,(k - r, N)$, $\nu = \mathrm{mod}\,(\sigma_s - \sigma_s' + k, N)$, and $\mathcal{J}(t_p)$ is a Laurent polynomial in t_p.

The case $j = 2$

One case in which we shall be particularly interested is when

$$k = 0, \quad \ell = 2, \quad j = 2. \tag{26}$$

For this case the functions $\eta, f_{vp}(j, \alpha, m)$ are given in (3.48) of ref. 8. From (16),

$$U(a, b, c, d) = \frac{N}{y_v y_{v'}} \left(\frac{b_p}{d_p} \right)^{a + d - b - c} \hat{U}(a, b, c, d), \tag{27}$$

where $\hat{U}(a, b, c, d)$ is given in Table I.

In particular, from (19) and (23) it follows that

$$\mathcal{I}(0, \dots, 0) = (y_v y_{v'} - \omega^{d - b + 1} t_p)^{s - 1}, \tag{28}$$

ignoring factors independent of p.

Table I. The Face Weights $\hat{U}(a,b,c,d)$ of the $\tau_2(t_p)$ Model

$a-d$	$b-c$	$\hat{U}(a,b,c,d)$
0	0	$y_v y_{v'} - \omega^{d-b+1} t_p$
0	1	$-\omega \mu_{v'} t_p (y_v - \omega^{d-b+1} x_{v'})$
1	0	$\mu_v (y_{v'} - \omega^{d-b+1} x_v)$
1	1	$-\omega \mu_v \mu_{v'} (t_p - \omega^{d-b+1} x_v x_{v'})$

The case $j = N$

The other case we shall need is when

$$k = -1, \qquad \ell = N+1, \qquad j = N. \tag{29}$$

Then $\zeta_{k\ell}$ is the full set of N integers $1, \ldots, N$ and $V_{vpq}(a, d; n)$, $V_{v,q,p}(-c, -b; n)$ are always given by (21), with α, β, m in the range $[0, N-1]$.

In this case, we shall be interested in the modified product (24), when $\sigma_s = \sigma_s' = 0$. Then $J_r(\sigma_1, \ldots, \sigma_s')$ is given by (25), with $v = N-1$. The factor $(d_p/b_p)^v$ is now independent of r and the spins $\sigma_1, \ldots, \sigma_{s-1}'$. Like the type (i) factors, it cancels out of the ratios of interest (34) and (47). The function $\mathcal{J}_r(\sigma_1, \ldots \sigma_s')$ in (25) is a polynomial in t_p of degree $(s-1)(N-1)$.

One particular sub-case that we shall consider is when $\sigma_1 = \cdots = \sigma_s' = 0$. From (3.26), (3.37) and (3.38) of ref. 8 we can verify that

$$\eta_{p,N,\alpha} = t_p^\alpha. \tag{30}$$

$$f_{vp}(N, N-1, m) = (d_v/b_v)^{N-1} x_v^m \tag{31}$$

and hence from (15) and (21) that

$$U(0, 0, 0, 0) = \prod_{j=1}^{N-1} (x_v x_{v'} - \omega^j t_p)^{s-1}, \tag{32}$$

ignoring factors independent of p. It follows that

$$\mathcal{J}_r(0,\ldots,0) = \prod_{j=1}^{N-1}(x_v x_{v'} - \omega^j t_p)^{s-1}. \qquad (33)$$

4. THE GENERALIZED ORDER PARAMETER

Now let us replace all the vertical rapidities p in Fig. 1 by v. We also replace all horizontal rapidities q by h, except for the one line immediately below the spin a. Following Jimbo et al.,[19] we break this line immediately below the site i containing the spin a, and give the half-line to the left the rapidity p, that to the right the rapidity q, as indicated in Fig. 4. Let $F_{pq}(a)$ be the probability that the spin at site i is in state a, i.e. from (10),

$$F_{pq}(a) = \langle \delta_{\sigma_i,a} \rangle = Z(a)/Z, \qquad (34)$$

where $Z(a)$ is the sum-over-states with spin σ_i fixed to be a, divided by the unrestricted sum $Z = Z(0) + \cdots + Z(N-1)$.
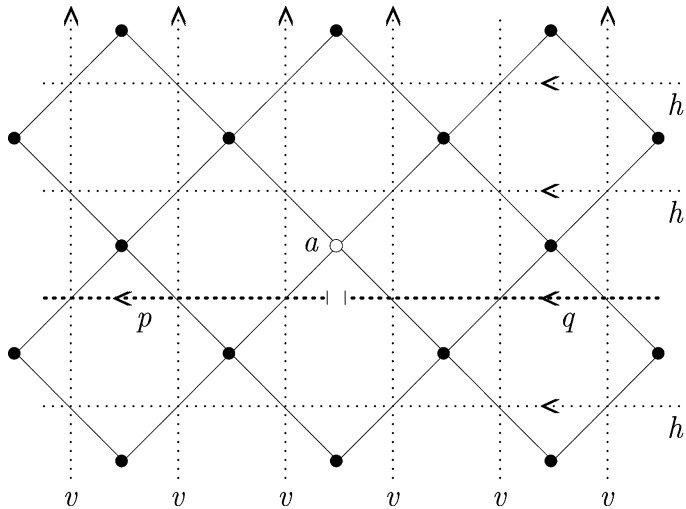

Fig. 4.   First picture of the function $F_{pq}(a)$.

Because of the star-triangle relation, $F_{pq}(a)$ is independent of the background rapidities v and h, but it does depend on p and q. This is because the ends of the half-lines near a cannot be moved away from a.

In ref. 20 we show that one can obtain functional relations satisfied by $F_{pq}(a)$. Let R, S be the automorphisms defined in ref. 5:

$$\{a_{Rp}, b_{Rp}, c_{Rp}, d_{Rp}\} = \{b_p, \omega a_p, d_p, c_p\}, \tag{35}$$

$$\{a_{Sp}, b_{Sp}, c_{Sp}, d_{Sp}\} = \{\omega^{-1/2} c_p, d_p, a_p, \omega^{-1/2} b_p\}, \tag{36}$$

so

$$x_{Rp} = y_p, \quad y_{Rp} = \omega x_p, \quad \mu_{Rp} = 1/\mu_p, \tag{37}$$

$$x_{Sp} = 1/y_p, \quad y_{Sp} = 1/x_p, \quad \mu_{Sp} = \omega^{-1/2} y_p/(x_p \mu_p). \tag{38}$$

Then from Eqs. (18) and (19) of ref. 5,

$$W_{pq}(n) = \bar{W}_{q, Rp}(-n), \quad \bar{W}_{pq}(n) = W_{q, Rp}(n),$$

$$W_{Rp, Rq}(n) = W_{pq}(-n), \quad \bar{W}_{Rp, Rq}(n) = \bar{W}_{pq}(-n), \tag{39}$$

$$W_{Sp, Sq}(n) = W_{qp}(n), \quad \bar{W}_{Sp, Sq}(n) = \bar{W}_{qp}(-n),$$

$$W_{Sp, RSq}(n) = \bar{W}_{pq}(-n), \quad \bar{W}_{Sp, RSq}(n) = W_{pq}(-n). \tag{40}$$

4.1. Symmetries

From (39), operating by R on all rapidities is equivalent to negating all spins. Hence

$$F_{Rp,Rq}(a) = F_{pq}(-a). \tag{41}$$

There is a reflection symmetry that is not given in ref. 20. Operate by S on all the vertical rapidities, and by RS on all the horizontal rapidities in Fig. 4. Then from (40) this is equivalent to interchanging the functions W, $\bar{W}$, and negating all spins. Interchanging W with $\bar{W}$ is in turn equivalent to mirror-reflecting the lattice about the central vertical line through a, while interchanging p with q. Thus

$$F_{RSp,RSq}(a) = F_{qp}(-a). \tag{42}$$

The dotted rapidity lines in Figs. 1 and 4 are directed, bearing arrows that give their direction. They form a graph $\mathcal{G}$ (the square lattice) of coordination number 4. The dual of G is a bi-partite graph, and one of the two sub-graphs is the lattice $\mathcal{L}$.

Although one cannot remove the ends of the half-lines from the spin a, one can rotate them subject to the rules given in ref. 20, notably that one is not allowed to introduce any directed circuits into $\mathcal{G}$. The effect of this is to deform $\mathcal{G}$, but the sites of $\mathcal{L}$ continue to live on one of the two sub-lattices dual to $\mathcal{G}$. Every edge of $\mathcal{L}$ passes through the intersection of two dotted rapidity lines of $\mathcal{G}$. If the two arrows on $\mathcal{G}$ lie on either side of the edge, with rapidities p, q, oriented as for the edge (j, k) of Fig. 1, then the weight function is $\bar{W}_{pq}(k - j)$. If both arrows lie on one side of the edge, oriented as for the edge (i, j), then the weight function is $W_{pq}(k - j)$.

We show in Fig. 8 of ref. 20 that if the arrows of all rapidity lines crossing a given line of rapidity p point to the left (right), one may reverse the arrow on p and replace p by Rp ($R^{-1}p$).

The result is that one can perform the following sequence of operations:

(a)   rotate the left half-line p clockwise through $90°$ to a vertical position below a, pointing downwards

(b)   replace p by $R^{-1}p$ and reverse the arrow to point upwards,

(c)   rotate this half-line $R^{-1}p$ clockwise though another $90°$ to the horizontal position of Fig. 5.
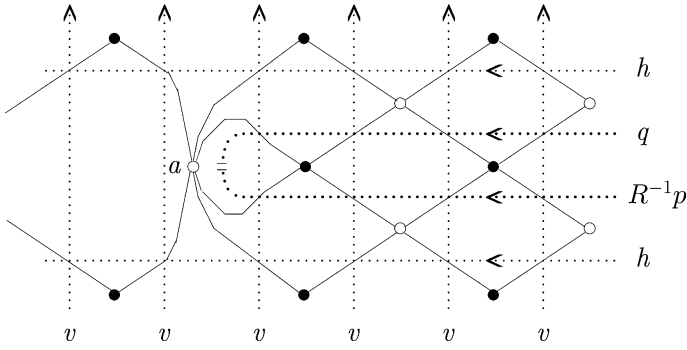
Fig. 5.   Second picture of the function $F_{pq}(a)$.

The probability $F_{pq}(a)$ is unchanged by this sequence, so we can alternatively use Fig. 5 to define $F_{pq}(a)$. We can interchange the rapidity half-lines $q, R^{-1}p$ therein without changing $F_{pq}(a)$, so $F_{pq}(a) = F_{p'q'}(a)$, where $q' = R^{-1}p$, $R^{-1}p' = q$. Hence

$$F_{pq}(a) = F_{Rq, R^{-1}p}(a), \tag{43}$$

which is the relation (15) of ref. 20.

Alternatively, one can similarly rotate and reverse the left half-line anti-clockwise round and above a to lie above q, with p replaced by Rp and again with both arrows pointing to the left. This is the configuration of Fig. 6. It too can be used for the definition of $F_{pq}(a)$.

Now if we pass the half-line Rp under q we obtain a different figure, where a is connected to the other sites only via a single edge of $\mathcal{L}$, to a site b to the right of a. This edge passes though the intersection of the
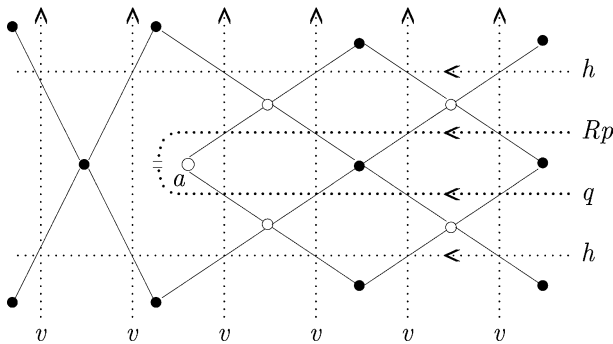

Fig. 6.   Third picture of the function $F_{pq}(a)$.

half lines Rp, q and has weight $\bar{W}_{Rp,q}(b-a)$. If we remove it, we regain Fig. 6, but with q interchanged with Rp and a replaced by b. Hence

$$F_{pq}(a) = \xi_{pq} \sum_{b=0}^{N-1} \bar{W}_{Rp,q}(b-a)\, F_{R^{-1}q, Rp}(b), \qquad (44)$$

where ξ_{pq} is independent of a. With a slight change of ξ_{pq}, this is the relation (16) of ref. 20.

4.2. Domains

If x_p, x_q, y_p, y_q and ωx_p all lie on the unit circle in an anti-clockwise ordered sequence, then the functions $W_{pq}(n)$, $\bar{W}_{pq}(n)$ are real and positive. This is the case we have in mind in this paper: it ensures that κ and the generalized order parameters $G_{pq}(r)$ exist in the thermodynamic limit of a large lattice and are continuous and infinitely differentiable.

For other values of p, q we define the functions to be the analytic continuations from this physical regime. To fix our ideas, it is helpful to consider the low-temperature case, when k' is small. If μ_p is of order k', then x_p is free to take most values in the complex plane, other than those near the N roots of unity $1, \omega, \ldots, \omega^{N-1}$, while y_p has to be near such a root of unity. If μ_p is of order k'^{-1}, then it is y_p that is free, x_p that is constrained. In Table II we show four such "domains". They form part of an increasing sequence: if p is in domain $\mathcal{D}_r$, then Rp is in domain $\mathcal{D}_{r+1}$.

Table II. Some Physical Rapidity Domains

Domain	x_p	y_p	μ_p^N
$\mathcal{D}_{-1}$	ω^{-1}	free	$O(k'^{-1})$
$\mathcal{D}_0$	free	1	$O(k')$
$\mathcal{D}_1$	1	free	$O(k'^{-1})$
$\mathcal{D}_2$	free	ω	$O(k')$

We also introduce the concept of a "bridge" $\mathcal{B}_r$ linking adjacent domains $\mathcal{D}_r$ and $\mathcal{D}_{r+1}$. On such a bridge μ_p^N is greater than $O(k')$ and less than $O(k'^{-1})$, and x_p, y_p are constrained by both $\mathcal{D}_r$ and $\mathcal{D}_{r+1}$. For instance, if p is on the bridge $\mathcal{B}_1$, then $x_p \simeq 1$ and $y_p \simeq \omega$.

If p is in $\mathcal{D}_r$ and q is on the bridge $\mathcal{B}_r$, then $W_{pq}(n)$, $\bar{W}_{pq}(n)$ are small for $n \neq 0$. The same is true if p is on $\mathcal{B}_r$ and q is in $\mathcal{D}_{r+1}$. The dominant contribution to the partition function is then from all spins being zero. One can develop the usual series expansions for κ for any such case: the results are all analytic continuations from the physical regime. We extend our terminology by referring to these cases as "physical".

An adjacent "near-physical" case is when $p, q \in \mathcal{D}_r$. Then $W_{pq}(n) = O(1)$, $\bar{W}_{pq}(n) = O(k'^{\alpha})$, where $\alpha = 2 \bmod (N - r, N)/N$ if r is odd, $\alpha = 2 \bmod (r, N)/N$ if r is even.[4] The $W_{pq}(n)$, $\bar{W}_{pq}(n)$ are therefore no bigger than $W_{pq}(0)$, $\bar{W}_{pq}(0)$. If this occurs on only one or two rapidity lines (or half-lines), while the other (background) rapidities are as in the previous paragraph, then we still expect the thermodynamic limit to exist.

The other "near-physical" case is when $p \in \mathcal{D}_r$, $q \in \mathcal{D}_{r+1}$. This has the same behaviour, except with $W_{pq}(n)$ interchanged with $\bar{W}_{pq}(n)$. Again we expect the thermodynamic limit to exist if this occurs on only one or two rapidity lines.

We focus attention on the low-temperature case when k' is small. However, we do not expect any discontinuities or non-analyticities for $0 < k' < 1$, so expect the above remarks to generalize to $0 < k' < 1$. For instance in $\mathcal{D}_1$ we take $|\mu_p| > 1$. Then x_p is constrained to a near-circular region enclosing the point $x_p = 1$ with $|\arg(x_p)| < \pi/(2N)$, while y_p lies anywhere in the complex plane except N such regions surrounding the points $1, \omega, \ldots, \omega^{N-1}$,[22] (Fig. 3).

We do need to consider whether the pictures of Figs. 4–6 all correspond to the same analytically continuous function. For the physical regime (all Boltzmann weights positive real), if $p \in_r$, then q should be in domains $\mathcal{D}_r$, $\mathcal{D}_{r-1}$ and $\mathcal{D}_{r+1}$, respectively. Then the thermodynamic limits of the partition functions of each figure will exist. They will be analytic continuations of one another, but will lie on different Riemann sheets.

Here we want to keep p, q lying in $\mathcal{D}_1$ for all three figures. For Fig. 4, this is the obvious analytic continuation of the physical case. We must ask whether we expect the partition functions of the other two figures to converge, and if so whether they will analytic continuations of the first.

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[4]To derive these formulae, one may have to use the alternative but equivalent form (59) of (8b).

The answer to both questions is yes. We start with

$$p, q, h \in \mathcal{D}_1, \quad v \in \mathcal{B}_0. \tag{45}$$

Consider the process outlined above for obtaining Fig. 5. When one rotates and reverses the half-line p to the lower vertical position, it then intersects the background horizontal lines with edge weight functions $W_{p',q'}, \bar{W}_{p',q}$, where $p' = R^{-1}p, q' = h$. Thus $p' \in \mathcal{D}_0, q' \in \mathcal{D}_1$. This is the second of the near-physical cases discussed above. The thermodynamic limit will still exist.

We then shift the background rapidities v, h one half-step down to $\mathcal{D}_0, \mathcal{B}_0$. The background weights remain physical, as are now those on the half-line $R^{-1}p$. Now those on q have weights $W_{vq}, \bar{W}_{vq}$, where $v \in \mathcal{D}_0$, $q \in \mathcal{D}_1$. This is the same near-physical case, so the thermodynamic limit still exists.

Finally we rotate the line $R^{-1}p$ through one more right-angle to assume the horizontal position of Fig. 5. Then the weights on it become $W_{v,R^{-1}p}, \bar{W}_{v,R^{-1}p}$, and $v, R^{-1}p \in \mathcal{D}_0$. This is the first of the near-physical cases, so again the thermodynamic limit exists.

Thus we can go continuously from the $F_{pq}(a)$ of Fig. 4 to that of Fig. 5, keeping the partition function convergent. We therefore expect both figures to give the same analytic function.

A similar argument applies to the rotations and reversals necessary to go from Figs. 4 to 6. Now we shift v, h one half-step up to $\mathcal{D}_1, \mathcal{B}_1$ after reversing p to Rp. Again, the background weights remain physical, while those on the two half-lines are either physical or near-physical. We expect both figures to give the same analytic function.

4.3. The Fourier Transform Ratio $G_{pq}(r)$

As in ref. 20, we define

$$\tilde{F}_{pq}(r) = \sum_{a=0}^{N-1} \omega^{ra} F_{pq}(a), \tag{46}$$

$$G_{pq}(r) = \tilde{F}_{pq}(r) / \tilde{F}_{pq}(r-1). \tag{47}$$

Then the above Eqs. (41)–(44) yield

$$G_{Rp,Rq}(r) = 1/G_{pq}(N-r+1), \tag{48}$$

$$G_{RSp,RSq}(r) = 1/G_{qp}(N-r+1), \tag{49}$$

$$G_{pq}(r) = G_{Rq,R^{-1}p}(r), \tag{50}$$

$$G_{pq}(r) = \frac{c_p a_q - a_p c_q \, \omega^r}{b_p d_q - d_p b_q \, \omega^{r-1}} \, G_{R^{-1}q,Rp}(r) \tag{51}$$

and from (47),

$$\prod_{r=1}^{N} G_{pq}(r) = 1. \tag{52}$$

Also,

$$G_{Mp,q}(r) = G_{p,M^{-1}q}(r) = G_{pq}(r+1), \tag{53}$$

where M is the rapidity operator such that

$$\{a_{Mp}, b_{Mp}, c_{Mp}, d_{Mp}\} = \{a_p, \omega^{-1}b_p, \omega^{-1}c_p, d_p\}. \tag{54}$$

A significant point that we shall use is that if $q = p$, then the break in the rapidity line in Fig. 4 disappears: the two half-lines become one, and then they *can* be removed to infinity. In the thermodynamic limit the probability $F_{pp}(a)$ is therefore independent of p, so it is also true that

$$G_{pp}(r) = \text{independent of } p. \tag{55}$$

Since $F_{pp}(a)$ is the probability that the central spin has value a,

$$\mathcal{M}_r = \langle \omega^{ra} \rangle = \tilde{F}_{pp}(r)/\tilde{F}_{pp}(0) = G_{pp}(1)\ldots G_{pp}(r), \tag{56}$$

so a knowledge of $G_{pq}(r)$ is certainly sufficient to determine the order parameter $\mathcal{M}_r$.

Equations (48)–(53) can be regarded as functional relations for the functions $G_{pq}(r)$. In ref. 20 we showed how they can be solved (using an analyticity assumption) for the Ising case $N = 2$. We also showed how an obvious generalization of this result to $N > 2$ satisfies the functional relations for $G_{pq}(r)G_{Rq,Rp}(r)$, but is *wrong*. (It disagrees with the known series expansions for $\mathcal{M}_r$.) This is a salutary lesson that these relations do not by themselves determine $G_{pq}(r)$: one has to input the correct analyticity properties.

5. LOW-TEMPERATURE LIMIT

Now we briefly consider the low-temperature limit, when k' is small, using the picture of Fig. 4.

As in refs. 20 and 26, we focus on the case when $p, q \in \mathcal{D}_1$, when

$$x_p, x_q \simeq 1, \quad \mu_p, \mu_q = O(k'^{-1/N}), \tag{57}$$

so y_p, y_q are arbitrary, of order unity. We also take $v \in \mathcal{B}_0$, $h \in \mathcal{D}_1$. This includes the physical regime where all the Boltzmann weights are real and positive.

The dominant contribution to $Z(a)$ then comes from the configuration where all spins other than a are zero. Then

$$F_{pq}(a) = W_{vp}(N-a)\,\bar{W}_{v'q}(N-a)\,\bar{W}_{vh}(a)\,W_{v'h}(a), \tag{58}$$

allowing the two vertical rapidities to have different values v, v'. Since $\bar{W}_{pq}(N) = 1$, we can divide the relation (8b) by $\bar{W}_{pq}(N)$ to obtain

$$\bar{W}_{pq}(n) = (\mu_p \mu_q)^{n-N} \prod_{j=n+1}^{N} \frac{y_q - \omega^j y_p}{\omega x_p - \omega^j x_q}. \tag{59}$$

Using this, (8a) and (7) (with $k=1$), we find that $F_{pq}(0) = 1$, while for $1 \leqslant a \leqslant N-1$,

$$F_{pq}(a) = \frac{k'^2 (\mu_p/\mu_q)^a}{N^2(1-\omega^a)(1-\omega^{-a})} \prod_{j=1}^{a} \frac{1-\omega^{j-1} t_q}{1-\omega^{j-1} t_p}. \tag{60}$$

This satisfies the relation (42).

As expected, the background rapidities v, v', h cancel out of these expressions. Also, when $q = p$, $F_{pq}(a)$ is independent of p. From (46),

$$\langle \omega^{ra} \rangle = \frac{\tilde{F}_{pp}(r)}{\tilde{F}_{pp}(0)} = 1 - k'^2 r(N-r)/(2N^2) + O(k'^4), \tag{61}$$

in agreement with (3) and (11). (The second-order terms in $F_{pq}(0)$ cancel out of this calculation.)

For $N = 3$ the author has obtained series expansions for $F_{pq}(a)$ to order k'^8 (ref. 26, Eq. (48)).

6. A SOLVABLE SPECIAL CASE

We have not determined $G_{pq}(r)$ for general values of the two half-line rapidities p and q. What we have done is obtain it for the case when $\{a_q, b_q, c_q, d_q\} = \{a_p, \omega b_p, c_p, d_p\}$, i.e.

$$x_q = x_p, \qquad y_q = \omega y_p, \qquad \mu_q = \mu_p. \tag{62}$$

The calculation and the result are very similar to the calculation of the free energy of the $\tau_2(t_q)$ model.

We have given the expressions (5) for the Boltzmann weights in terms of the original rapidity variables a_p, b_p, c_p and d_p because they make it clear that the weights remain finite and non-zero when b_p, b_q become zero. This corresponds to taking $y_p = y_q = 0$. Hence the special case (62) then intersects the desired case $q = p$. From (55), this is sufficient to determine $G_{pp}(r)$ and hence the order parameter $\mathcal{M}_r$.

In this section, we shall consider the three pictures in Figs. 4–6 of the function $F_{pq}(a)$. They all give the same function $F_{pq}(a)$ in the limit of an infinite lattice. However, they do differ for a finite lattice. We shall use this difference to manifest different properties of $F_{pq}(a)$.

We shall then put these properties together to make what we believe to be a plausible and correct assumption as to the analyticity properties of the Fourier transform ratio $G_{pq}(r)$ for $0 < k' < 1$. This assumption is the key to derivation of this paper.

One function that we shall use is

$$\epsilon(r) = 1 - N\delta_{r,0}, \tag{63}$$

where $\delta_{r,0}$ is to be interpreted modulo N. Thus $\epsilon(0) = \epsilon(N) = 1 - N$, and $\epsilon(r) = 1$ for $r = 1, \ldots, N - 1$.

6.1. First Picture

Consider the definition (34) of $F_{pq}(a)$, where $Z(a)$ is the partition function of the lattice shown in Fig. 4. For the moment ignore the restrictions (62) and allow p, q to be general rapidities. For a finite lattice (with boundary spins fixed to zero), it is readily seen that

$$F_{pq}(a) = (\mu_p/\mu_q)^a \times (\text{rational function of } x_p, y_p, x_q, y_q). \tag{64}$$

Consider μ_p, x_p, y_p as functions of the complex variable t_p. From (7) they are multi-valued functions, with N branch cuts $\mathcal{C}_0, \ldots, \mathcal{C}_{N-1}$. The cut
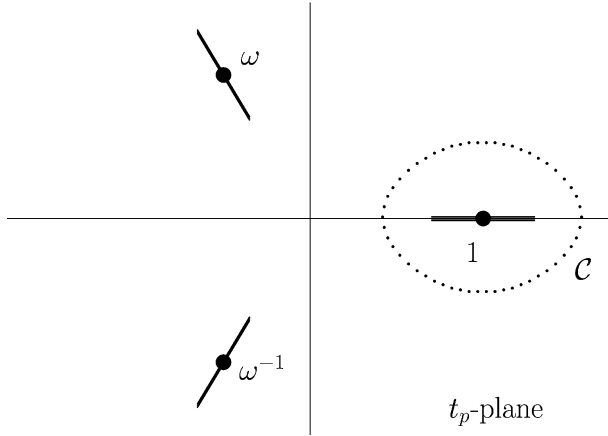
Fig. 7.   The cut t_p-plane for $N = 3$, showing the contour $\mathcal{C}$.

$\mathcal{C}_i$ is along the straight-line segment $(\omega^i \rho, \omega^i / \rho)$, where $\rho = [(1 - k')/(1 + k')]^{1/N}$. For $N = 3$, these cuts are shown in Fig. 7. On them $|\mu_p| = 1$. Here we concentrate our attention on the case where $p \in \mathcal{D}_1$, which is when

$$|\mu_p| > 1, \qquad |\arg x_p| < \pi/(2N). \tag{65}$$

For k' small, x_p is restricted to a small region $\mathcal{R}$ round $x_p = 1$. We shall say $x_p \in \mathcal{R}$, or simply $x_p \simeq 1$. On the other hand, y_p can lie almost anywhere in the complex plane, being excluded only from $\mathcal{R}$ and corresponding small regions round $y_p = \omega, \ldots, \omega^{N-1}$.

For a finite lattice, it follows from (64) that $F_{pq}(a)$ is also a multi-valued function of t_p. To make it single-valued, we must restrict t_p to the cut plane Fig. 7.

Now consider the low-temperature limit of Section 5. Making the substitutions (62) into (60), we observe that there are significant cancellations. We obtain

$$F_{pq}(a) = \frac{k'^2}{N^2(1 - \omega^a)(1 - \omega^{-a})} \frac{1 - \omega^a t_p}{1 - t_p} + \mathrm{O}(k'^4). \tag{66}$$

From (46), (47) it follows that

$$G_{pq}(r) = 1 - \frac{(N - 1 - 2r)k'^2}{2N^2} - \frac{\epsilon(r)\, k'^2}{N^2(1 - t_p)} + \mathrm{O}(k'^4)$$

for $r = 0, \ldots, N - 1$.

To this order there is no evidence of any zero or singularity in $G_{pq}(r)$ for any value of t_p other than one. This singularity is consistent with there being a branch cut on the real axis, as in Fig. 7, but at this very preliminary stage there is no evidence that the other $N-1$ branch cuts, or any other singularities, occur.

6.2. Second Picture

Now we consider the picture of Fig. 5 for $Z(a)$ and write

$$p' = R^{-1}p \tag{67}$$

for the rapidity of the lower half-row. The rapidity of the upper half-row is q.

Then $x_p = y_{p'}$, $y_p = \omega x_{p'}$, $\mu_p = 1/\mu_{p'}$, so from (62) the rapidities of the two rows are related by

$$x_q = y_{p'}, \quad y_q = \omega^2 x_{p'}, \quad \mu_q = 1/\mu_{p'}. \tag{68}$$

For the moment, take the lattice to be finite, of half-width s. In Fig. 5, we have shown the sites of $\mathcal{L}$ immediately above and below the two half-rapidity lines (including the left-hand site with spin a) as open circles. All other sites are shown as filled circles.

If we duplicate the spin a as two open circles at the same heights as the others, these open circle spins correspond precisely to the spins in Fig. 3. The spins between them lie at the centre of stars, as in Fig. 2. Summing over them, we obtain precisely the weight (19), with $\sigma_1, \sigma'_1 = a$. Because the boundary spins are fixed to zero, we also have $\sigma_s, \sigma'_s = 0$.

Comparing (68) with (13), we see that we have

$$k=0, \qquad \ell=2, \qquad j=2, \tag{69}$$

so we can use the results of the $j=2$ Section 3 for the contribution to each term in $Z(a)$ of edges crossing the half-lines $R^{-1}p, q$. Thus $Z(a)$ is a sum of expressions (19) (with coefficients independent of p, q), so from (23) $Z(a)$ *is a polynomial in* t_p *of degree* $s-1$. It has no other dependence on p.

Again we consider the low-temperature case (57). The dominant contribution to any $Z(a)$ comes from the configuration where all the remaining spins other than a are zero. Also, the other four edges incident to spin

a give a weight that is unity when $a=0$, while when $a\neq 0$ to leading order their contribution is $k'^2/[N^2(1-\omega^a)(1-\omega^{-a})]$.

Using Table I, with p replaced by p' and $y_v=y_{v'}=1$, and noting from the equations of this section that $\omega t_{p'}=t_p$, it follows that

$$Z(0) = (1-t_p)^{s-1}+O(k'^2),$$

and for $a\neq 0$

$$Z(a) = \frac{k'^2(1-\omega^a t_p)(1-t_p)^{s-2}}{N^2(1-\omega^a)(1-\omega^{-a})}+O(k'^4). \tag{70}$$

Again we obtain the leading-order result (66) for $F_{pq}(a)$, but now we see also that, for s finite and $a\neq 0$,

$$\tilde{F}_{pq}(a) = \tilde{Z}(a)/\tilde{Z}(0), \tag{71}$$

where the $\tilde{Z}(a)$ are polynomials in t_p of degree $s-1$, equal to $(1-t_p)^{s-1}$ when $k'=0$.

Hence, by continuity, for sufficiently small k' the zeros of the polynomials $\tilde{Z}(a)$ must lie close to $t_p=1$.

As we take the limit of s large, we only expect singularities to occur in the vicinity of these zeros. This reinforces the suggestion of the last sub-section that in the limit of a large lattice, $\tilde{F}_{pq}(a)$ and $G_{pq}(r)$ may be analytic functions of t_p except for singularities in some region surrounding $t_p=1$. Again, there is no evidence of the branch cuts in Fig. 7, other than the cut on the real positive axis.

6.3. Third Picture

Finally we consider the picture of Fig. 6 for $Z(a)$, and write

$$p'=q, \qquad q'=Rp \tag{72}$$

for the rapidities of the half-rows immediately below and above the spin a.

Then $x_{q'}=y_p$, $y_{q'}=\omega x_p$, $\mu_{q'}=1/\mu_p$, so from (62) the rapidities of the two rows are related by

$$x_{q'}=\omega^{-1}y_{p'}, \qquad y_q=\omega x_{p'}, \qquad \mu_q=1/\mu_{p'}. \tag{73}$$

Again we initially take the lattice to be finite, of half-width s. The open circle spins in Fig. 6, other than a, correspond to the spins in Fig. 3. Let the lower such spins be $\sigma_1, \dots, \sigma_s$, the upper ones $\sigma'_1, \dots, \sigma'_s$. Summing over the spins between them, we obtain the weight (19), with the boundary spins $\sigma_s, \sigma'_s = 0$. We also obtain the weight of the two edges incident to a. From (46) and (17), these contribute a further weight $V_{vq'p'}(-\sigma'_1, -\sigma_1; -r)$ to $\tilde{F}_{pq}(r)$. Using (24), we see that the total factor contributed by the edges that cross the lines q, Rp is $J_r(\sigma_1, \dots \sigma'_s)$.

From (73),

$$k = -1, \qquad \ell = N+1, \qquad j = N, \tag{74}$$

so we can use the results of the last Section 3, with $\nu = N - 1$. Summing over all the spins other than a in Fig. 6, we see from (25) that

$$x_p^{m(r)} \tilde{F}_{pq}(r) = \mathcal{A}_r / \mathcal{A}_0, \tag{75}$$

where

$$m(r) = \mathrm{mod}\,(N - r - 1, N) \tag{76}$$

and each $\mathcal{A}_r$ is a weighted sum of functions $\mathcal{J}_r$, with coefficients independent of p. Thus $\mathcal{A}_r$ is a polynomial in t_p of degree $(N-1)(s-1)$. This is the complete dependence on p.

When $k' = 0$, then $x_v = x_{v'} = 1$ and the only contribution to the sum-over-states is from all spins being zero, so from (33)

$$\mathcal{A}_r = \prod_{i=1}^{N-1} (1 - \omega^i t_p)^{s-1}. \tag{77}$$

From continuity, for sufficiently small k' we see that all the zeros of the polynomials $\mathcal{A}_r$ must be clustered round the Nth roots of unity in the complex t_p-plane, *excluding* $t_p = 1$.

So now we have information that is reciprocal to that of Section 6. In the thermodynamic limit of s large we expect $x_p^{m(r)} \tilde{F}_{pq}(r)$ to have singularities near the $t_p = \omega, \omega^2, \dots, \omega^{N-1}$, but *not* near $t_p = 1$. From (75) and (47),

$$G_{pq}(r) = x_p^{\epsilon(r)} \mathcal{A}_r / \mathcal{A}_{r-1} \tag{78}$$

for $r = 0, \dots, N - 1$.

6.4. The Relation (51)

So far we have not used any of the six functional relations (48)–(53). Half of them are not helpful for the present argument, since if the arguments p, q of the function G on one side of the relation satisfy our restriction (62), then those on the other side do not.

One relation that is helpful is (51). Set

$$p_1 = R^{-1} q, \qquad q_1 = R p, \tag{79}$$

then, using (35) and (62), we obtain

$$x_{q_1} = x_{p_1}, \qquad y_{q_1} = \omega y_{p_1}, \qquad \mu_{q_1} = \mu_{p_1}, \tag{80}$$

so p_1 and q_1 also satisfy the restriction (62). The relation (51) becomes

$$G_{pq}(r) = \frac{x_p(1 - \omega^r)}{y_p(1 - \omega^r)} G_{p_1, q_1}(r), \tag{81}$$

which implies

$$x_p^{-1} G_{pq}(r) = y_p^{-1} G_{p_1, q_1}(r) \tag{82}$$

for $r = 1, \ldots, N - 1$. Also, from (52) it follows that

$$x_p^{-\epsilon(r)} G_{pq}(r) = y_p^{-\epsilon(r)} G_{p_1, q_1}(r) \tag{83}$$

for $r = 0, \ldots, N - 1$.

If we write $p = \{x_p, y_p, \mu_p\}$, then $p_1 = \{y_p, x_p, 1/\mu_p\}$. Hence p_1 is obtained from p by interchanging x_p with y_p and inverting μ_p, which leaves t_p interchanged. This is what one obtains by taking t_p across the branch cut on the positive real axis in Fig. 7 and then returning it to its original value. The relations (82) and (83) therefore say that the functions on the lhs of the equations also return to their original values, i.e. that they do *not* have this branch cut in the complex t_p plane. This agrees with (78) and our observations that we do not expect the ratios of the A_r to have any singularities in the vicinity of $t_p = 1$.

7. ANALYTICITY ASSUMPTIONS

Define, for $r = 1, \ldots, N$,

$$H_{pq}(r) = x_p^{-\epsilon(r)} G_{pq}(r). \tag{84}$$

The functions $G_{pq}(r)$ are defined by (46) and (47) in terms of $F_{pq}(a)$. This in turn is proportional to $Z(a)$, the partition function of the lattice with the broken rapidity line. Three equivalent forms of this lattice are shown in Figs. 4–6.

In Section 6 we took q to be related to p by (62). We presented evidence based on Figs. 4 and 5 to suggest that for sufficiently small k' the functions $G_{pq}(r)$ are analytic and non-zero in the complex t_p-plane, excepting some region surrounding and not far from the point $t_p = 1$. Then we used Fig. 6 to obtain evidence suggesting that the functions $H_{pq}(r)$ are analytic and non-zero except near $t_p = \omega, \ldots, \omega^{N-1}$. We showed that this is consistent with the functional relation (51).

We remarked at the beginning of Section 5 that the chiral Potts Boltzmann weight functions $W_{pq}(n)$, $\bar{W}_{pq}(n)$ remain finite when $b_q = 0$ and $a_q, c_q, d_q \neq 0$. For this reason we neither see nor expect any singularities in the $G_{pq}(r)$ at $t_p = 0$. The same is true when $c_q = 0$, implying that the $G_{pq}(r)$ remain finite and analytic when $t_p \to \infty$.

We therefore make the following analyticity assumptions, taking p, q to be related by (62), and regarding $G_{pq}(r), H_{pq}(r)$ as functions of the complex variable t_p. We take both p and q to lie in the domain $\mathcal{D}_1$, i.e. to satisfy (65). Although our evidence is obtained when k' is sufficiently small, we do not expect any non-analyticities in k' throughout the ferromagnetic regime $0 < k' < 1$. We therefore expect the assumptions to be true for $0 < k' < 1$.

ASSUMPTIONS

There exists a closed contour $\mathcal{C}$ surrounding the branch cut on the positive real axis in Fig. 7, such that

(i)   the functions $G_{pq}(r)$ are analytic, bounded and non-zero outside and on $\mathcal{C}$.

(ii)   the functions $H_{pq}(r)$ are analytic and non-zero inside and on $\mathcal{C}$.

The functional relation (51) implies that $H_{pq}(r)$ has no branch cut on the positive real axis, so (ii) follows if we strengthen (i) to apply to the whole t_p-plane except for the cut on the positive real axis.

These assumptions are very similar to those we made to calculate the free energy of the $\tau_2(t_q)$ model by the "inversion relation" method (ref. 22, p. 419). In fact we shall find that $G_{pq}(r)$ has a similar form to that free energy.

These analyticity properties can be expressed as symmetries.[22,27] For instance, if we exhibit the dependence of $G_{pq}(r)$ on x_p, y_p by writing it as $G_r(x_p, y_p)$, then Assumption (i) implies

$$G_r(x_p, y_p) = G_r(\omega^j y_p, \omega^{-j} x_p)$$

for $j = 1, \ldots, N-1$, $(x_p, y_p) \in \mathcal{D}_1$ and $(\omega^j y_p, \omega^{-j} x_p)$ in a domain adjacent to $\mathcal{D}_1$.

8. CALCULATION OF $G_{pq}(r)$

The above assumptions are sufficient to calculate each of $G_{pq}(0), \ldots, G_{pq}(N-1)$ to within some constant factor. These factors can then be determined from the property (55).

Then we can write (84) as

$$x_p^{\epsilon(r)} = G_{pq}(r)/H_{pq}(r). \tag{85}$$

Then it follows immediately from the above Assumption that we can obtain the functions $G_{pq}(r), H_{pq}(r)$ by a Wiener–Hopf factorization of $x_p^{\epsilon(r)}$. (See for example Eqs. (50)–(53) of ref. 22.)

The variables x_p, μ_p are function of t_p: let us write them as $x(t_p)$, $\mu(t_p)$. The function $\log x(t_p)$ is single-valued and analytic on $\mathcal{C}$ in the t_p-plane. Define

$$B_-(t) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\log x(s)}{s-t} \, ds, \tag{86}$$

where the integration is now round the corresponding curve $\mathcal{C}$ in the complex s-plane, and t lies outside $\mathcal{C}$. Define $B_+(t)$ by the same equation, but taking t to be inside $\mathcal{C}$. In the first case shrink $\mathcal{C}$ to a curve $\mathcal{C}_-$ just inside $\mathcal{C}$, in the second expand it to $\mathcal{C}_+$ lying just outside $\mathcal{C}$. Now take t to lie between $\mathcal{C}_+$ and $\mathcal{C}_-$. Then the combined curve of integration for $B_+(t) - B_-(t)$ can be deformed to a closed curve surrounding t and close to $\mathcal{C}$, so by Cauchy' integral formula

$$B_+(t) - B_-(t) = \log x(t). \tag{87}$$

Hence from (85),

$$\epsilon(r)B_-(t) + \log G_{pq}(r) = \epsilon(r)B_+(t) + \log H_{pq}(r). \tag{88}$$

From (86) and our assumption, the lhs is analytic and bounded outside $\mathcal{C}$. Similarly, the rhs is analytic and bounded inside $\mathcal{C}$. Each side is therefore entire and bounded, so be Liouville's theorem each is a constant. Hence

$$\log G_{pq}(r) = \text{ constant } - \epsilon(r)B_-(t_p), \tag{89}$$

where the constant is independent of p, but may depend on r.

This result is of course consistent with our assumption, and with the stronger assumption that $G_{pq}(r)$ is analytic and non-zero in the t_p-plane, except for the cut along the real axis in Fig. 7. The contour $\mathcal{C}$ can be shrunk to just surround this cut.

Integrating by parts, we can write (86) as

$$B_-(t_p) = = -\frac{1}{2\pi i} \oint_{\mathcal{C}} \log(s - t_p) \frac{x'(s)}{x(s)} ds, \tag{90}$$

The quantity $\mu(s)^N$ moves once around the unit circle as s moves around $\mathcal{C}$ (in the positive directions). Setting $\mu(s) = \exp(-i\theta/N)$ and using (89), we obtain for $r = 1, \ldots, N$,

$$G_{pq}(r) = C_r \, S(t_p)^{\epsilon(r)}, \tag{91}$$

where C_r is a constant,

$$\log S(t_p) = -\frac{2}{N^2} \log k + \frac{1}{2N\pi} \int_0^{2\pi} \frac{k' e^{i\theta}}{1 - k' e^{i\theta}} \log[\Delta(\theta) - t_p] d\theta, \tag{92}$$

and

$$\Delta(\theta) = [(1 - 2k' \cos\theta + k'^2)/k^2]^{1/N}. \tag{93}$$

The $\log k$ term has been included in (92) to ensure that $S(0) = 1$, choosing the logarithms to ensure that $S(t_p)$ is positive real when t_p is real and off the branch cut. The condition (52) is satisfied provided that

$$C_1 C_2 \cdots C_N = 1. \tag{94}$$

8.1. Calculation of $C_1, ..., C_N$

We still need to determine the constants $C_1, \ldots, C_N$. We noted at the beginning of Section 5 that when $y_q = y_p = 0$, with the restriction (62), $G_{pq}(r) = G_{pp}(r)$. In this case $t_p = 0$ and we can evaluate the integral in (91), giving $S(0) = 1$ and

$$G_{pp}(r) = C_r. \tag{95}$$

Now consider the case when $a_q, \ldots, d_p$ satisfy the restrictions given immediately before (62), and $c_q = c_p = 0$, so $y_q = y_p = \infty$. It is not true that $q = p$, but from (54) it is true that $q = M^{-1}p$. Now $t_q = t_p = \infty$ and the integral in (92) is zero. Using (53), we therefore have

$$G_{pp}(r+1) = G_{p,M^{-1}p}(r) = C_r \, k^{-2\epsilon(r)/N^2}. \tag{96}$$

Because of (55), the $G_{pp}(r)$ in these last two equations are the same. Eliminating them, we obtain

$$C_{r-1} = k^{2/N^2} C_r \tag{97}$$

for $r = 2, \ldots, N$. From these equations and (94), it follows that

$$C_r = k^{(N+1-2r)/N^2} \tag{98}$$

for $r = 1, \ldots, N$.

This completes the calculation of $G_{pq}(r)$ for the case when p, q are related by (62): it is given by (91) and (98).

We also immediately obtain from (95) that

$$G_{pp}(r) = k^{(N+1-2r)/N^2} \tag{99}$$

for $r = 1, \ldots, N$. Hence from (56),

$$\mathcal{M}_r = \langle \omega^{ra} \rangle = k^{r(N-r)/N^2} \tag{100}$$

for $r = 0, \ldots, N$. This verifies the conjecture (3).

8.2. Relation to $\tau_2(t_p)$

The free energy of the $\tau_2(t_p)$ model is given in Eq. (39) of ref. 10 and (73) of ref. 22. These expressions are very similar to the integral in (91). In fact if we use Eq. (73) of ref. 22 for $\tau_2(t_p)$, and manifest its dependence on λ_q by writing it as $\tau_2(t_p, \lambda_q)$, then

$$k^{1/N} S(\omega t_p)^{2N} = \tau_2(t_p, 0)/\tau_2(t_p, k'). \tag{101}$$

8.3. The product $G(t_p, r) \cdots G(\omega^{N-1} t_p, r)$

Exhibit the dependence of $G_{pq}(r)$ on t_p by writing it as $G_r(t_p)$. Consider the product

$$L_r(t_p) = G_r(t_p) G_r(\omega t_p) \cdots G_r(\omega^{N-1} t_p). \tag{102}$$

This can be evaluated from (91), giving:

$$L_r(t_p) = D_r x_p^{\epsilon(r)}, \tag{103}$$

where

$$D_r = C_r^N k^{-\epsilon(r)/N} = k^{1+\delta_{r,N}-2r/N} \tag{104}$$

for $r = 1, \dots, N$.

In fact we could have obtained this result very directly from the assumptions of Section 6. These imply that the functions $x_p^{-\epsilon(r)} G_r(t_p)$, $G_r(\omega t_p), \dots, G_r(\omega^{N-1} t_p)$ are all analytic on and near the positive real axis. Hence so is the product $L_r(t_p)/x_p^{\epsilon(r)}$. Since this is unchanged by multiplying t_p by ω, none of the branch cuts in Fig. 7 appear, so it is analytic everywhere. It is bounded, so by Liouville's theorem it is a constant. This verifies (103), but does not determine the constant. To do this, repeat the argument of (95)–(98). From (7), $x_p = k^{1/N}$ when $y_p = 0$, and $x_p = k^{-1/N}$ when $y_p = \infty$, so we obtain

$$G_{pp}(r)^N = D_r k^{\epsilon(r)/N} = D_{r-1} k^{-\epsilon(r-1)/N}. \tag{105}$$

From (52), $D_1 D_2 \cdots D_N = 1$. It follows that D_r is indeed given by (104). It also follows that $G_{pp}(r)$ is given by (99), which verifies the conjecture (3). This route avoids using the Wiener–Hopf factorization and the integral (91).

9. OTHER SPECIAL CASES

It is natural to ask if we can evaluate $G_{pq}(r)$ when p, q are related, not by (62), but by the more general condition

$$x_q = x_p, \qquad y_q = \omega^i y_p, \qquad \mu_q = \mu_p, \tag{106}$$

where $i = 0, \ldots, N - 1$.

For such a case, let us introduce an extra index i and write

$$G_{pq}(r) = G_{ir}(t_p), \qquad L_{ir}(t_p) = \prod_{j=0}^{N-1} G_{ir}(\omega^j t_p). \tag{107}$$

As a first step, one can use the series (48) of ref. 26 to expand $L_{ir}(t_p)$ to order k'^8 for $N = 3$. Indeed, this is how the author discovered that the case (62) might be tractable. For $i = 2$ and $r = 0, 2$ we have not observed anything particularly simple, but for $i = 2, r = 1$ the series are consistent with the conjectures

$$L_{21}(t_p) = x_p^2, \qquad G_{21}(t_p) = k^{2/9} S(t_p) S(\omega t_p), \tag{108}$$

$S(t_p)$ being the function defined by (92).

We can use the general result (60) to expand $G_{ir}(t_p)$ to first order in k'^2. For $i = 0, \ldots, N - 1$ and $r = 1, \ldots, N - i$ the results simplify, giving

$$G_{ir}(t_p) = 1 + \frac{k'^2}{2N^2} \left(2r + 2i - N - 1 - 2 \sum_{j=1}^{i} \frac{1}{1 - \omega^{j-1} t_p} \right),$$

which is consistent with the formula

$$G_{ir}(t_p) = k^{(N+1-2r)/N^2} S(t_p) S(\omega t_p) \cdots S(\omega^{i-1} t_p). \tag{109}$$

This formula also agrees with (99) when $t_p = 0, q = p$, and when $t_p = \infty, q = M^{-i} p$. From preliminary calculations we suspect (109) can be justified in the same way that we justified Assumptions (i) and (ii), by arguing that $G_{pq}(r)$ has no branch cuts on $\mathcal{C}_1, \ldots, \mathcal{C}_{N-i}$, while $H_{pq}(r)$ has no branch cuts on $\mathcal{C}_0, \mathcal{C}_{-1}, \ldots, \mathcal{C}_{1-i}$. We could then derive (109) by a Wiener–Hopf factorization, taking the contour $\mathcal{C}$ to surround the cuts $\mathcal{C}_0, \mathcal{C}_{-1}, \ldots, \mathcal{C}_{1-i}$.

For other values of i, r awkward factors such as $1/(1-\omega)$ occur that multiply the $1/(1-\omega^{j-1} t_p)$ terms, and the conditions to use (21) in (24) are no longer satisfied because $n = -r$ is not in $\zeta_{k\ell}$. We have observed no simple patterns for these cases.

10. SUMMARY

We have derived the long-conjectured result (3), (11) for the order parameters of the ferromagnetic chiral Potts model. The method does depend on the analyticity assumptions of Section 7, but in this respect it is no different from the standard derivation of the chiral Potts model free energy. In both calculations one argues that a certain "$\tau_2(t_p)$" type-function is analytic except for a single branch cut, and obtains a rule relating the two values of the function on either side of that cut.[9,10,22]

In one respect this order parameter calculation is easier than that of the free energy. Although we have performed a Wiener–Hopf factorization to obtain the function $S(t_p)$ of Section 8, we did not need to. As we remarked at the end of that section, it is sufficient to calculate the elementary function $L_r(t_p)$.

It seems that it is this function that is the desired generalization to arbitrary N of the simple $N = 2$ function $L_{pq}(r)$ of ref. 20. For $N = 2$ the chiral Potts model reduces to the Ising model. There is a difference property and all functions depend on p, q only via k and $u_q - u_p$, where u_p, u_q are elliptic function arguments. For a long time the author thought one had to keep p, q independent to avoid the trivial situation when $u_q - u_p$ is merely a constant. Of course this is not so: for $N = 2$ the restriction (62) corresponds to $u_q = 2K - u_p$, K being a constant (the elliptic integral). Hence $u_q - u_p = 2K - 2u_p$ and the functions do not merely degenerate to constants. It is this "superintegrable" $N = 2$ case that we have generalized in this paper.

The actual calculation is not difficult, being given in Section 8. Most of this paper is concerned with presenting arguments for the analyticity assumptions of Section 7.

For $N = 3$, low-temperature expansions of $G_{pq}(r)$ were developed in ref. 26. As we remarked above, these were very useful in developing and checking the ideas that led to our analyticity assumptions. They agree with the results (91), (98), and with the conjecture (109).

We have only obtained the generalized order parameter function $G_{pq}(r)$ when p, q are related by (106), with $i = 0$ and 1. At the end of Section 9 we have also conjectured how these results may extend to $i = 2, \ldots, N - 1$, $r = 1, \ldots, N - i$. It would be of some interest to study the

other cases of (106), and indeed the general case of no restriction on p, q. This last is not an easy problem: we intend to comment on the difficulties in a subsequent paper.

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